Seminar “PI-Algebras and Related Topics”
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The Amitsur-Levitzki theorem, its proofs and applications

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Let K be an arbitrary field of any characteristic and let $M_n(K)$ be the algebra of $n \times n$ matrices with entries from K. Let also $K\langle X\rangle = K\langle x_1, x_2, \ldots\rangle$ be the free unitary associative algebra freely generated by the set $X = \{x_1, x_2, \ldots\}$, i.e. the algebra of polynomials in infinitely many noncommuting variables.

Amitsur-Levitzki theorem

The matrix algebra $M_n(K)$ satisfies the standard identity of degree $2n$

$$s_{2n} = \sum_{\sigma \in S_{2n}} \operatorname{sign}(\sigma) x_{\sigma(1)} \cdots x_{\sigma(2n)} = 0,$$

where S_{2n} is the symmetric group of degree $2n$. With exception of the case $n \leq 2$ and $K = GF_2$, up to a multiplicative constant this is the only polynomial identity of minimal degree for the matrix algebra. In the exceptional case $M_n(GF_2)$, $n \leq 2$, satisfies a nonlinear identity of the same minimal degree $2n$.

The Amitsur-Levitzki theorem is one of the most famous theorems and a corner stone in the theory of algebras with polynomial identity (PI-algebras). It has several proofs based on different arguments.

- S.A. Amitsur, J. Levitzki, Minimal identities for algebras, Proc. Amer. Math. Soc. **1** (1950), 449-463.
- B. Kostant, A theorem of Frobenius, a theorem of Amitsur-Levitzki and cohomology theory, J. Math. Mech. **7** (1958), 237-264.
- R.G. Swan, An application of graph theory to algebra, Proc. Amer. Math. Soc. **14** (1963), 367-373. Correction: **21** (1969), 379-380.
- Yu.P. Razmyslov, Trace identities of full matrix algebras over a field of characteristic zero (Russian), Izv. Akad. Nauk SSSR, Ser. Mat. **38** (1974), 723-756. Translation: Math. USSR, Izv. **8** (1974), 727-760.
- S. Rosset, A new proof of the Amitsur-Levitzki identity, Israel J. Math. **23** (1976), 187-188.
- J. Szigeti, Z. Tuza, G. Révész, Eulerian polynomial identities on matrix rings, J. Algebra **161** (1993), No. 1, 90-101.
- C. Procesi, On the theorem of Amitsur-Levitzki, Isr. J. Math. **207** (2015), Part 1, 151-154.

The purpose of the talk is to discuss the proofs of the Amitsur-Levitzki theorem, to give some of them and to present some consequences of the theorem.

The talk is based also on the books

- V. Drensky, Free Algebras and PI-Algebras, Springer-Verlag, Singapore, 2000.
- V. Drensky, E. Formanek, Polynomial Identity Rings, Advanced Courses in Mathematics, CRM Barcelona, Birkhäuser, Basel-Boston, 2004.

and the papers

- Amitsur-Levitzki theorem, in Encyclopedia of Mathematics and Wikipedia

The proof of Amitsur and Levitzki

The original proof of the theorem uses the following facts:

- Since the standard polynomial $s_{2n}(x_1,\ldots,x_{2n})$ is linear in each argument, it is sufficient to show that it vanishes when we replace the variables with the matrix units e_{ij}, $i,j=1,\ldots,n$.
- Since $s_{2n}(x_1,\ldots,x_{2n})$ is skew symmetric, i.e.

$$s_{2n}(x_{\tau(1)},\ldots,x_{\tau(2n)})=\mathrm{sign}(\tau)s_{2n}(x_1,\ldots,x_{2n}),\ \tau\in S_{2n},$$

  it vanishes when two of the arguments are equal.

It is based on inductive combinatorial arguments. With some technical improvements it can be found for example in the book by Passmann (p. 175).

- D.S. Passman, The Algebraic Structure of Group Rings, Wiley-Interscience, New York, 1977.

Proof of the uniqueness. In 1950 Levitzki proved that the identity of minimal degree for $M_n(K)$ is of degree $\geq 2n$. His proof is a consequence of more complicated results on PI-algebras. See Consequence 2 in his paper.

- J. Levitzki, A theorem on polynomial identities, Proc. Am. Math. Soc. **1** (1950), 334-341.

It is known that if an algebra R satisfies a polynomial identity of degree d, then it satisfies a multilinear polynomial identity of degree d. Here we give a direct proof of the uniqueness in the case of multilinear identities. Let $M_n(K)$ satisfy the multilinear identity of degree $d \leq 2n$

$$f(x_1, \ldots, x_d) = \sum_{\sigma \in S_d} a_\sigma x_{\sigma(1)} \cdots x_{\sigma(d)} = 0, \, a_\sigma \in K.$$

First, let $d < 2n$. Let for example $a_\varepsilon \neq 0$, where ε is the identical permutation of S_d. (The cases for the other a_σ are handled in a similar way.) We replace $x_1, x_2, x_3, x_4, \ldots, x_d$ by the matrix units $r_1 = e_{11}, r_2 =_{12}, r_3 =_{22}, r_4 = e_{23}$, etc. The only nonzero product is $r_1 r_2 \cdots r_d = e_{1k}$. Hence $f(r_1, \ldots, r_d) = a_\varepsilon e_{1k} \neq 0$ and the case $d < 2n$ is impossible.

Now, let $d = 2n$ and

$$f(x_1, \ldots, x_{2n}) = \sum_{\sigma \in S_{2n}} a_\sigma x_{\sigma(1)} \cdots x_{\sigma(2n)}.$$

We evaluate f on

$$r_1 = r_2 = e_{11}, r_3 = e_{12}, r_4 = e_{22}, r_5 = e_{23}, \ldots, r_{2n-1} = e_{n-1,n}, r_{2n} = e_{nn}.$$

Then $F(r_1, \ldots, r_{2n}) = (a_\varepsilon + a_{(12)})e_{1n} = 0$. Hence $a_{(12)} = -a_\varepsilon$. With similar arguments we obtain that if σ and τ are two permutations such that $\sigma(i) = \tau(i)$ for all i different from k and $k+1$ and $\sigma(k) = \tau(k+1)$, $\sigma(k+1) = \tau(k)$, then $a_\tau = -a_\sigma$. Since every permutation σ can be obtained from the identical by consecutive exchanging the places of the numbers at two adjacent positions, we obtain that $a_\sigma = \text{sign}(\sigma)a_\varepsilon$ and

$$f(x_1, \ldots, x_{2n}) = a_\varepsilon s_{2n}(x_1, \ldots, x_{2n}).$$

Now, let $K = GF_2$ be the field with two elements.

- It is well known that $M_1(GF_2) = GF_2$ *satisfies the identity* $x^2 + x = 0$.
- *The algebra* $M_2(GF_2)$ *satisfies the identity*

$$f(x, y) = xy^3 + yxy^2 + y^2xy + y^3x + xy^2 + y^2x = 0.$$

Proof. Over GF_2 there are four quadratic polynomials only:

$$x^2,\ x^2+1,\ x^2+x,\ x^2+x+1.$$

By the Cayley-Hamilton theorem for every matrix $a \in M_2(GF_2)$ one of the following holds:

$$a^2=0,\ a^2=e,\ a^2=a,\ a^2=a+e.$$

Let $a^2=\alpha a+\beta e$, $\alpha,\beta\in GF_2$. Since $\alpha^2=\alpha$, we obtain

$$a^3=a(\alpha a+\beta e)=\alpha(\alpha a+\beta e)+\beta a=(\alpha+\beta)a+\beta e.$$

Replacing a^3 and a^2 in $f(x,a)$ we obtain immediately $f(x,a)=0$, i.e. $f(x,y)=0$ is an identity for $M_2(GF_2)$.

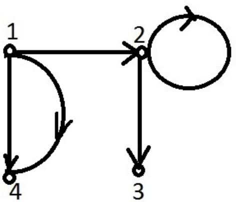
The proof of Kostant

The proof is cohomological and depends upon the Frobenius theory of representations of the alternating group. The paper by Kostant was also the first to relate the polynomial identities satisfied by matrices with traces, a theme which was later developed by Procesi and Razmyslov and influenced much research.

It is interesting to mention that all of the proofs of the Amitsur-Levitzki theorem except the original combinatorial proof and the graph theoretical proofs of Swan and of Szigeti, Tuza and Révész depend on the Cayley-Hamilton Theorem.

The proof of Swan

The main idea of the proof is to define a correspondence between any set of matrix units e_{ij} (allowing repetitions) and an oriented graph (allowing loops and several edges with the same beginning and end) with a set of vertices $\{1, 2, \ldots, n\}$ and edges (ij) for any e_{ij}.



The graph in the picture corresponds to the set of matrix units $\{e_{12}, e_{14}, e_{14}, e_{22}, e_{23}\}$ and the path $1 \to 2 \to 2 \to 2 \to 3$ illustrates the product $e_{12}e_{22}e_{22}e_{23} = e_{13}$.

Let Γ be an oriented graph with sets of vertices $\{1, 2, \dots, n\}$ and edges $\{e_1, e_2, \dots, e_k\}$. Let $\sigma \in S_k$. If $(e_{\sigma(1)}, e_{\sigma(2)}, \dots, e_{\sigma(k)})$ is a path, it is called a *unicursal path* from the origin of $e_{\sigma(1)}$ to the end of $e_{\sigma(k)}$. Depending on the parity of σ, the path may be either even or odd. The idea of the proof of Swan is based on the following observation.

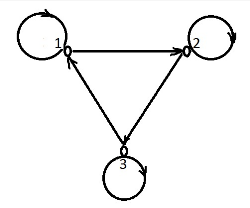
If $e_{i_1 j_1}, \dots, e_{i_{2n}, j_{2n}} \in M_n(K)$ are matrix units, and $e_1, \dots, e_{2n}$ are the edges of the corresponding oriented graph, then

$$s_{2n}(e_{i_1 j_1}, \dots, e_{i_{2n} j_{2n}}) = \sum_{i,j=1}^{n} (a_{ij}^{(+)} - a_{ij}^{(-)}) e_{ij},$$

where $a_{ij}^{(+)}$ and $a_{ij}^{(-)}$ are, respectively, the number of even and odd unicursal paths from i to j.

The main result in graph theory which immediately implies the Amitsur-Levitzki theorem is the following.

Theorem. *Let Γ be an oriented graph with a set of vertices V and a set of edges E. If $|E| \geq 2|V|$, then for any two vertices v_1 and v_2 the number of even unicursal paths from v_1 to v_2 is equal to the number of odd unicursal paths.*



If the matrix units are $e_{11}, e_{12}, e_{22}, e_{23}, e_{33}, e_{31}$ and the corresponding vertices of the graph are $e_1, e_2, e_3, e_4, e_5, e_6$, then all unicursal paths from 2 to 2 are $(e_3, e_4, e_5, e_6, e_1, e_2)$ which is even and $(e_4, e_5, e_6, e_1, e_2, e_3)$ which is odd.

The proof of Razmyslov

Lemma. *The validity of the Amitsur-Levitzki theorem for $M_n(\mathbb{Q})$ implies its validity for $M_n(K)$ over any field K.*

Proof. If $r_p = \sum_{i,j=1}^{n} \alpha_{ij}^{(p)} e_{ij}$, $\alpha_{ij}^{(p)} \in K$, $p = 1, \ldots, 2n$, are matrices in $M_n(K)$, then $s_{2n}(r_1, \ldots, r_{2n})$ is a linear combination of $s_{2n}(e_{i_1 j_1}, \ldots, e_{i_{2n} j_{2n}})$ and is equal to 0 because we have assumed that the theorem holds for $M_n(\mathbb{Z}) \subset M_n(\mathbb{Q})$.

Lemma. *Let the eigenvalues of the matrix $a \in M_n(K)$ be equal to $\xi_1, \dots, \xi_n$ and let $e_q(\xi_1, \dots, \xi_n)$ be the q-th elementary symmetric polynomial in $\xi_1, \dots, \xi_n$. Then*

$$a^n + \sum_{q=1}^{n} (-1)^q e_q(\xi_1, \dots, \xi_n) a^{n-q} = 0,$$

$$\mathrm{tr}(a^q) = \xi_1^q + \cdots + \xi_n^q.$$

Proof. The characteristic polynomial $f(\lambda) = \det(\lambda e - a)$ of a is the same as for its Jordan normal form

$$\begin{pmatrix} \xi_1 & * & \cdots & * \\ 0 & \xi_2 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \chi_n \end{pmatrix}.$$

Hence

$$f(\lambda) = \prod_{i=1}^{n}(\lambda - \xi_i) = \lambda^n + \sum_{q=1}^{n}(-1)^q e_q(\xi_1, \ldots, \xi_n)\lambda^{n-q}$$

Now the proof follows from the Cayley-Hamilton theorem applied to the matrix a.

The proof of Razmyslov of the Amitsur-Levitzki theorem. We give the proof for 2×2 matrices. The general case is similar with additional technical difficulties only. By one of the lemmas we assume that $K = \mathbb{Q}$. Let

$$p_q(\xi_1, \dots, \xi_n) = \xi_1^q + \dots + \xi_n^q.$$

The Newton formulas give that for $q \leq n$

$$p_q - e_1 p_{q-1} + e_2 p_{q-2} + \dots + (-1)^{q-1} e_{q-1} p_1 + (-1)^q q e_q = 0.$$

We can express $e_q(\xi_1, \dots, \xi_n)$ as polynomials of $p_q(\xi_1, \dots, \xi_n)$. In our case

$$e_2(\xi_1, \xi_2) = \xi_1 \xi_2 = \frac{1}{2}((\xi_1 + \xi_2)^2 - (\xi_1^2 + \xi_2^2)) = \frac{1}{2}(p_1^2(\xi_1, \xi_2) - p_2(\xi_1, \xi_2)).$$

By the other lemma, for any 2×2 matrix a with eigenvalues ξ_1, ξ_2

$$a^2 - e_1(\xi_1, \xi_2)a + e_2(\xi_1, \xi_2)e = 0,$$

$$a^2 - p_1(\xi_1, \xi_2)a + \frac{1}{2}(p_1^2(\xi_1, \xi_2) - p_2(\xi_1, \xi_2))e = 0,$$

$$a^2 - \mathrm{tr}(a)a + \frac{1}{2}(\mathrm{tr}^2(a) - \mathrm{tr}(a^2))e = 0.$$

Hence

$$f(x) = x^2 - \mathrm{tr}(x)x + \frac{1}{2}(\mathrm{tr}^2(x) - \mathrm{tr}(x^2))e = 0$$

is a *trace polynomial identity* for $M_2(K)$.

We linearize this trace identity (i.e. obtain the consequence $f(y_1+y_2)-f(y_1)-f(y_2)$):

$$(y_1y_2+y_2y_1)-(\mathrm{tr}(y_1)y_2+\mathrm{tr}(y_2)y_1)$$

$$+\frac{1}{2}((\mathrm{tr}(y_1)\mathrm{tr}(y_2)+\mathrm{tr}(y_2)\mathrm{tr}(y_1))-\mathrm{tr}(y_1y_2+y_2y_1))e=0.$$

Since

$$\mathrm{tr}(y_1)\mathrm{tr}(y_2)=\mathrm{tr}(y_2)\mathrm{tr}(y_1),\ \mathrm{tr}(y_1y_2)=\mathrm{tr}(y_2y_1),$$

we see that $M_2(K)$ satisfies the multilinear trace identity

$$g(y_1,y_2)=(y_1y_2+y_2y_1)-(\mathrm{tr}(y_1)y_2+\mathrm{tr}(y_2)y_1)$$

$$+(\mathrm{tr}(y_1)\mathrm{tr}(y_2)-\mathrm{tr}(y_1y_2))e=0.$$

Now we replace y_1 and y_2 respectively by $x_{\sigma(1)}x_{\sigma(2)}$ and $x_{\sigma(3)}x_{\sigma(4)}$ and take the alternating sum on $\sigma \in S_4$:

$$0 = \sum_{\sigma \in S_4} \operatorname{sign}(\sigma) f(x_{\sigma(1)}x_{\sigma(2)}, x_{\sigma(3)}x_{\sigma(4)})$$

$$= 2 \sum_{\sigma \in S_4} \operatorname{sign}(\sigma)(x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}x_{\sigma(4)} - \operatorname{tr}(x_{\sigma(1)}x_{\sigma(2)})x_{\sigma(3)}x_{\sigma(4)})$$

$$+ \sum_{\sigma \in S_4} \operatorname{sign}(\sigma)(\operatorname{tr}(x_{\sigma(1)}x_{\sigma(2)})\operatorname{tr}(x_{\sigma(3)}x_{\sigma(4)}) - \operatorname{tr}(x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}x_{\sigma(4)})e).$$

The trace is invariant under cyclic permutations, hence

$$\mathrm{tr}(x_{\sigma(1)}x_{\sigma(2)}) = \mathrm{tr}(x_{\sigma(2)}x_{\sigma(1)}),$$

$$\mathrm{tr}(x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}x_{\sigma(4)}) = \mathrm{tr}(x_{\sigma(2)}x_{\sigma(3)}x_{\sigma(4)}x_{\sigma(1)}).$$

The permutations in each of the pairs

$$(\sigma(1),\sigma(2),\sigma(3),\sigma(4)) \text{ and } (\sigma(2),\sigma(1),\sigma(3),\sigma(4)),$$

$$(\sigma(1),\sigma(2),\sigma(3),\sigma(4)) \text{ and } (\sigma(2),\sigma(3),\sigma(4),\sigma(1))$$

are of different parity and the summands containing traces vanish in

$$\sum_{\sigma\in S_4} \mathrm{sign}(\sigma) g(x_{\sigma(1)}x_{\sigma(2)}, x_{\sigma(3)}x_{\sigma(4)}) = 0.$$

Therefore, we obtain that

$$2s_4 = 2\sum_{\sigma\in S_4}(\mathrm{sign}\sigma)x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}x_{\sigma(4)} = 0,$$

and this completes the proof.

The proof of Rosset

For the proof of Rosset we need some knowledge about the exterior (or the Grassmann) algebra $E(V)$, where V is a vector space with basis $\{v_i \mid i = 1, 2, \dots\}$. It is the associative algebra generated by the basis of V with defining relations

$$v_i v_j + v_j v_i = 0,\ v_i^2 = 0,\ i, j = 1, 2, \dots.$$

Then $E(V)$ has a basis

$$\{v_{i_1} \cdots v_{i_k} \mid i_1 < \cdots < i_k,\ k = 0, 1, 2, \dots\}$$

and the elements $v_{i_1} \cdots v_{i_k}$ of even length span the center of $E(V)$.

In a similar way one defines the Grassmann algebra $E(V_m)$ when V_m is an m-dimensional vector space.

Lemma.

Let K be a field of characteristic 0 and let for the matrix $a \in M_n(K)$

$$\mathrm{tr}(a^k) = 0,\ k = 1, 2, \ldots, n.$$

Then $a^n = 0$.

Proof. As in the proof of Razmyslov, if $\xi_1, \ldots, \xi_n$ are the eigenvalues of the matrix a, then

$$a^n + \sum_{q=1}^{n} (-1)^q e_q(\xi_1, \ldots, \xi_n) a^{n-q} = 0.$$

By the Newton formulas we can express $e_q(\xi_1, \ldots, \xi_n)$ in terms of

$$p_k(\xi_1, \ldots, \xi_n) = \xi_1^k + \ldots + \xi_n^k,\ k = 1, \ldots, q,$$

i.e. in terms of tr(a^k), $k = 1, \ldots, q$. Since all tr$(a^k) = 0$, we obtain that $a^n = 0$.

Lemma.

Let C be a commutative ring. Then for any $a_1, \dots, a_{2k} \in M_n(C)$

$$\mathrm{tr}(s_{2k}(a_1, \dots, a_{2k})) = 0.$$

Proof. By definition $s_{2k}(a_1, \dots, a_{2k})$ is a sum of products $\pm a_{\sigma(1)} \cdots a_{\sigma(2k)}$, $\sigma \in S_{2k}$. We subdivide these products in groups obtained by cyclic permutations of a given product. Then all of the monomials in one group have the same trace, but half have a plus sign and half have a minus sign. Hence $\mathrm{tr}(s_{2k}(a_1, \dots, a_{2k})) = 0$.

The proof of Rosset of the Amitsur-Levitzki theorem. As we have seen, it is sufficient to prove the theorem when K is a field of characteristic 0. Let V_{2n} be a $2n$-dimensional vector space with basis $\{v_1, \dots, v_{2n}\}$ and let $E(V_{2n})$ be the Grassmann algebra on V_{2n}. Then the algebra D generated by the products $v_i v_j$, $1 \leq i < j \leq 2n$, is commutative. Consider the matrix algebra $M_n(E(V_{2n}))$ with entries from $E(V_{2n})$. For any $a_1, \dots, a_{2n} \in M_n(K)$ let $b = a_1 v_1 + \cdots + a_{2n} v_{2n}$. Then

$$c = b^2 = \sum_{1 \leq i < j \leq 2n} (a_i a_j - a_j a_i) v_i v_j \in M_n(D),$$

and

$$c^k = b^{2k} = \sum s_{2k}(a_{i_1}, \dots, a_{i_{2k}}) v_{i_1} \cdots v_{i_{2k}}.$$

Since c belongs to the matrix algebra $M_n(D)$ with entries from the commutative ring D and by the lemma for $\text{tr}(s_{2k})$, we obtain that $\text{tr}(c) = \text{tr}(c^2) = \cdots = \text{tr}(c^n)$ and hence $c^n = 0$. But $c_n = s_{2n}(a_1, \dots, a_{2n}) v_1 \cdots v_{2n}$ and hence $s_{2n}(a_1, \dots, a_{2n}) = 0$.

The proof of Szigeti, Tuza and Révész

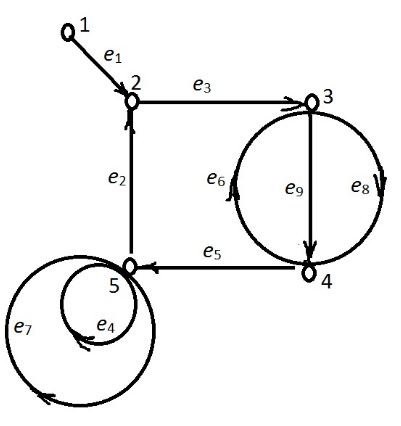
Szigeti, Tuza and Révész consider an oriented connected graph Γ with a vertex set $\{1,\dots,k\}$ and an edge set $\{e_1,\dots,e_N\}$ (allowing several edges with the same beginning and end), where each vertex i is a beginning of $\phi_+(i)$ and end of $\phi_-(i)$ edges. They fix two vertices p and q (it is allowed $p=q$) and call the graph Γ *Eulerian* if one of the following holds:

- $p=q$ and $\phi_+(i)-\phi_-(i)=0$ for all $i=1,\dots,k$;
- $p\neq q$ and $\phi_+(i)-\phi_-(i)=\begin{cases}0, & \text{if } i\neq p,q;\\ 1, & \text{if } i=p;\\ -1, & \text{if } i=q.\end{cases}$

Let $\Pi(\Gamma) \subset S_n$ be the set of all permutations corresponding to paths $(e_{\sigma(1)}, \dots, e_{\sigma(n)})$ of Γ from p to q. As in the proof of Swan such paths are unicursal.

A well known theorem in graph theory states:
Theorem. *If Γ is a connected oriented Eulerian graph with fixed points p and q, then there is a unicursal path from p to q.*

The idea of Szigeti, Tuza and Révész is to paraphrase the theorem of Swan:
Theorem. *Let Γ be an oriented graph with a set of vertices V and a set of edges E. If $|E| \geq 2|V|$, then for any two vertices v_1 and v_2 the number of even unicursal paths from v_1 to v_2 is equal to the number of odd unicursal paths.*



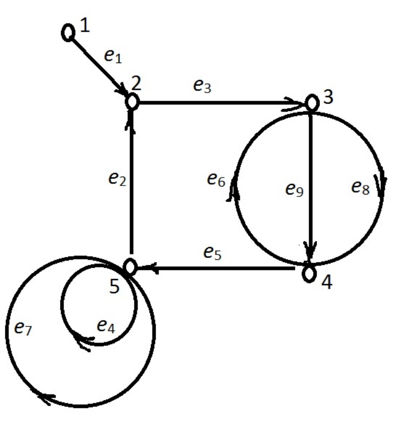
$p = 1$, $q = 2$. The paths are

even: $(e_1, e_3, e_9, e_6, e_8, e_5, e_4, e_7, e_2), (e_1, e_3, e_8, e_6, e_9, e_5, e_7, e_4, e_2)$,

odd: $(e_1, e_3, e_8, e_6, e_9, e_5, e_4, e_7, e_2), (e_1, e_3, e_8, e_6, e_9, e_5, e_4, e_7, e_2)$.

Let Γ be an Eulerian graph with vertices $\{1, \ldots, k\}$ and edges $\{e_1, \ldots, e_N\}$. Define

$$\gamma(i) = \begin{cases} \phi_+(i), & \text{if } i \neq q; \\ \phi_-(i), & \text{if } i = q. \end{cases}$$



$$\gamma(1) = 1, \gamma(2) = \gamma(3) = \gamma(4) = 2, \gamma(5) = 3.$$

The theorem of Szigeti, Tuza and Révész. *Let* Γ *be an Eulerian graph with* k *vertices and* N *edges. If* $n \geq 1$ *is such that*

$$N \geq 2 \sum_{i=1}^{k} \min\{n, \gamma(i)\},$$

then

$$P_\Gamma(x_1, \ldots, x_N) = \sum_{\sigma \in \Pi(\Gamma)} \mathrm{sign}(\sigma) x_{\sigma(1)} \cdots x_{\sigma(N)} = 0$$

is a polynomial identity for $M_n(K)$.

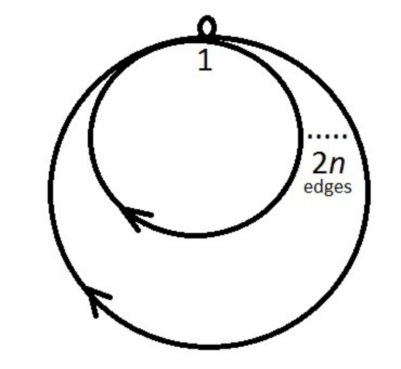
Corollary. *Let Γ be an Eulerian graph with k vertices and N edges. If $n \geq 1$ is such that $N \geq 2kn$, then $P_\Gamma(x_1, \ldots, x_N) = 0$ is a polynomial identity for $M_n(K)$.*

Proof. Since $n \geq \min\{n, \gamma(i)\}$, we obtain that

$$N \geq 2nk \geq 2 \sum_{i=1}^{k} \min\{n, \gamma(i)\}$$

and the corollary follows immediately from the theorem.

Special case — the standard identity:



$$k = 1,\ N = 2n,\ \Pi(\Gamma) = S_{2n}$$

$$P_\Gamma(x_1, \ldots, x_{2n}) = s_{2n}(x_1, \ldots, x_{2n}) = 0.$$

The proof of Procesi

Procesi shows that the Amitsur-Levitzki theorem is the Cayley-Hamilton theorem for the generic Grassmann matrix. Techinally, the proof is very similar to the proof of Rosset but while in the proof of Rosset the Grassmann variables are auxiliary, in the proof of Procesi these variables are intrinsically embedded in the problem; this is important for applications.

One of the main problems for the polynomial identities for matrices

Problem. Over a field K of characteristic 0 find a basis for the polynomial identities of $M_n(K)$.

Known answers

- $n = 1$ (trivial answer). All identities follow from

$$[x_1, x_2] = x_1 x_2 - x_2 x_1 = 0.$$

- $n = 2$. Razmyslov: *All polynomial identities for $M_2(K)$ follow from the standard identity $s_4(x_1, x_2, x_3, x_4) = 0$ and the identities of degree* 5.
  Drensky: *The polynomial identities for $M_2(K)$ follow from*

$$s_4(x_1, x_2, x_3, x_4) = [[x_1, x_2]^2, x_1] = 0.$$

- $n \geq 3$. Kemer: *The polynomial identities for any associative algebra follow from a finite number.*

- Yu.P. Razmyslov, Trace identities of full matrix algebras over a field of characteristic zero (Russian), Izv. Akad. Nauk SSSR, Ser. Mat. **38** (1974), 723-756. Translation: Math. USSR, Izv. **8**, 727-760 (1974).
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Other polynomial identities for matrices. The identity of algebraicity

The identity of algebraicity

$$a_n(x_1,\ldots,x_n,y)=\sum_{\sigma\in S_{n+1}}\mathrm{sign}(\sigma)y^{\sigma(0)}x_1y^{\sigma(1)}x_2y^{\sigma(2)}\cdots x_ny^{\sigma(n)}=0$$

is a polynomial identity for $M_n(K)$. For $n>1$ it does not follow from the standard identity $s_{2n}(x_1,\ldots,x_{2n})=0$. Here S_{n+1} acts on the set $\{0,1,\ldots,n\}$.

Proof. Consider the *Capelli identity*

$$c_{n+1}(y_0,y_1,\ldots,y_n;x_1,\ldots,x_n)=\sum_{\sigma\in S_{n+1}}\mathsf{sign}(\sigma)y_{\sigma(0)}x_1y_{\sigma(1)}\cdots\cdots x_ny_{\sigma(n)}.$$

It vanishes when $y_0,y_1,\ldots,y_n$ are linearly dependent. Replace y_i by y^i and apply the Cayley-Hamilton theorem.

Let $n > 1$ and let $a_n = 0$ follow from $s_{2n} = 0$. Then the same holds for

$$b_n(x,y) = a_n(x,\dots,x,1,y) = \sum_{\sigma\in S_{n+1}} \mathrm{sign}(\sigma) y^{\sigma(0)} x y^{\sigma(1)} \cdots\cdots x y^{\sigma(n)}.$$

Hence

$$b_n(x,y) = \sum \alpha_i u_i s_{2n}(v_{i1},\dots,v_{i,2n}) w_i,$$

where u_i, v_{ij}, w_i are monomials in x and y and

$$\deg_x(u_i) + \sum_{j=1}^{2n} \deg_x(v_{ij}) + \deg_x(w_i) = n,$$

$$\deg_y(u_i) + \sum_{j=1}^{2n} \deg_y(v_{ij}) + \deg_y(w_i) = 1 + 2 + \cdots + n.$$

It holds $s_{2n}(1, x_2, \ldots, x_{2n}) = 0$. (For the proof, we write $s_{2n}(x_1, \ldots, x_{2n})$ as a sum of products of commutators

$$\frac{\mathrm{sign}(\sigma)}{2^n}[x_{\sigma(1)}, x_{\sigma(2)}] \cdots [x_{\sigma(2n-1)}, x_{\sigma(2n)}].)$$

Hence, the only possibility is

$$u_i = w_i = 1, v_1 = \cdots = v_n = x, v_{n+1} = y, v_{n+2} = y^2, \ldots, v_{2n} = y^n$$

and this is impossible because $n \geq 2$ and $v_1 = v_2$, i.e. the evaluation of s_{2n} is equal to 0.

Conjecture. If $\mathrm{char}(K) = 0$, then for $n \geq 2$ the polynomial identities of $M_n(K)$ follow from the standard identity $s_{2n} = 0$ and the identity of algebraicity $a_n(x_1, \ldots, x_n, y) = 0$.

- True for $n = 2$, because $[[x, y]^2, y] = a_2(x, y)$.
- Not true for $n = 3$. Okhitin and Domokos found identities of degree 9 which do not follow from $s_6 = a_3 = 0$.

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The double Capelli identity

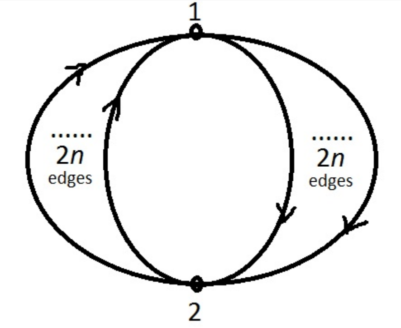
Problem. (Formanek) Let

$$d_k(X,Y)=\sum_{\sigma,\tau\in S_k}\mathrm{sign}(\sigma\tau)x_{\sigma(1)}y_{\tau(1)}\cdots x_{\sigma(k)}y_{\tau(k)}=0$$

be the double Capelli identity. Does it follow from the standard identity $s_k(X)=0$?

- The answer into affirmative was given by Chan.
- Giambruno and Sehgal showed that $d_{2n}(X,Y)=0$ is a polynomial identity for $M_n(K)$.
- Szigeti, Tuza and Révész deduced this from their graph theoretical approach.
- Domokos found an easy proof using the Amitsur-Levitzki theorem.

The proof of Szigeti, Tuza and Révész



$$k = 2,\ N = 4n,\ P_\Gamma(x_1,\ldots,x_{2n},y_1,\ldots,y_{2n}) = d_{2n}(X,Y).$$

The proof of Domokos

Let $a_1, \ldots, a_{2n}, b_1, \ldots, b_{2n} \in M_n(K)$. Consider the 2×2 block matrices

$$A_i = \begin{pmatrix} 0 & a_i \\ 0 & 0 \end{pmatrix}, B_i = \begin{pmatrix} 0 & 0 \\ b_i & 0 \end{pmatrix} \in M_{2n}(K),\ i = 1, \ldots, 2n.$$

By the Amitsur-Levitzki theorem for $M_{2n}(K)$

$$s_{4n}(A_1, B_1, \ldots, A_{2n}, B_{2n}) = \begin{pmatrix} 0 & d_{2n}(A, B) \\ d_{2n}(B, A) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence $d_{2n}(X, Y) = 0$ is a polynomial identity for $M_n(K)$.

THANK YOU VERY MUCH
FOR YOUR ATTENTION!